

Entropy operator and associated Wigner function

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Abstract

We show how the entropy operators for two subsystems may be calculated. In the case of the atom-field interaction we obtain the associated Wigner function for the entropy operator for the quantized field.

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I. INTRODUCTION

It is well known that when two subsystems A and B interact the entropy of the total system and the subsystem entropies obey a triangle inequality[1]

$$|S_A - S_B| \leq S_{AB} \leq S_A + S_B. \quad (1)$$

This result implies that if both subsystems are initially in pure states, the total entropy is zero and both subsystems entropies will be equal after they interact. Here we would like to arrive to the result that, if initially the two subsystems are in pure states *any* function the density matrix of subsystem A is equal to the function of the density matrix of subsystem B , and in particular obtain that both subsystems entropies are equal, without using the Araki-Lieb theorem. We would also like to find the entropy operator

$$\hat{S}_{A(B)} = -\ln \hat{\rho}_{A(B)}, \quad (2)$$

for any of the subsystems. In particular we will consider later a two-level atom-field interaction, but the results may be generalized to other kind of subsystems, for instance atom-atom interaction, atom-many atoms interaction, N -level atom-field interaction, etc.

The density matrix for a two-level system interacting with another subsystem B is given by

$$\hat{\rho} = \begin{pmatrix} |c\rangle\langle c| & |c\rangle\langle s| \\ |s\rangle\langle c| & |s\rangle\langle s| \end{pmatrix} \quad (3)$$

where $|c\rangle$ ($|s\rangle$) is the unnormalized wave function of the second system corresponding to the excited (ground) state of the two level system. From the total density matrix we may obtain the subsystem density matrices as

$$\hat{\rho}_A = \begin{pmatrix} \langle c|c\rangle & \langle s|c\rangle \\ \langle c|s\rangle & \langle s|s\rangle \end{pmatrix} \equiv \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \quad (4)$$

and

$$\hat{\rho}_B = |c\rangle\langle c| + |s\rangle\langle s|. \quad (5)$$

From the density matrix for subsystem B one can not see a clear way to calculate the entropy operator as powers of ρ_B get complicated to be obtained. To make the calculation easier we state that *if two subsystems are initially in pure states, after interaction, the trace of any*

function of one of the subsystems density matrix is equal to the trace of the function of the other subsystem's density matrix.

We can prove this, by using the following relation valid for two interacting systems that before interaction were in pure states

$$\hat{\rho}_B^{n+1} = Tr_A\{\hat{\rho}(t)\hat{\rho}_A^n(t)\}, \quad (6)$$

with this relation it is easy to show that

$$Tr_B\{\hat{\rho}_B^{n+1}(t)\} = Tr_A\{\hat{\rho}_A^{n+1}(t)\}. \quad (7)$$

In particular with the expression (6) we demonstrate that $S_B = S_A$. Of course it is also true that

$$\hat{\rho}_A^{n+1} = Tr_B\{\hat{\rho}(t)\hat{\rho}_B^n(t)\}. \quad (8)$$

II. PROVING $\hat{\rho}_B^{n+1} = Tr_A\{\hat{\rho}(t)\hat{\rho}_A^n(t)\}$ BY INDUCTION

We can prove relation (6) by induction. For this we need to find $\hat{\rho}_A^n$, to this end we write

$$\hat{\rho}_A = \frac{\mathbf{1}}{2} + \begin{pmatrix} \frac{\delta}{2} & \rho_{12} \\ \rho_{21} & -\frac{\delta}{2} \end{pmatrix} \equiv \frac{\mathbf{1}}{2} + \hat{R}, \quad (9)$$

where $\delta = \rho_{11} - \rho_{22}$ and $\mathbf{1}$ is the 2×2 unit density matrix. We can then find simply

$$\hat{\rho}_A^n = \left(\frac{\mathbf{1}}{2} + \hat{R}\right)^n = \sum_{m=0}^n \binom{n}{m} \frac{1}{2^{n-m}} \hat{R}^m. \quad (10)$$

We split the above sum into two sums, one with odd powers of \hat{R} and one with even powers. We also use that

$$\hat{R}^{2m} = \epsilon^{2m}, \quad \hat{R}^{2m+1} = \frac{\hat{R}}{\epsilon} \epsilon^{2m+1} \quad (11)$$

with $\epsilon = \left(\frac{\delta^2}{4} + |\rho_{12}|^2\right)^{1/2}$. Therefore we can write

$$\hat{\rho}_A^n = \frac{1}{2} \left[\left(\frac{1}{2} + \epsilon\right)^n + \left(\frac{1}{2} - \epsilon\right)^n \right] \mathbf{1} + \frac{\hat{R}}{2\epsilon} \left[\left(\frac{1}{2} + \epsilon\right)^n - \left(\frac{1}{2} - \epsilon\right)^n \right] \quad (12)$$

In terms of ρ_A the above equation is written as

$$\hat{\rho}_A^n = G(n)\hat{\rho}_A - \|\hat{\rho}_A\|G(n-1)\mathbf{1} \quad (13)$$

where

$$G(n) = \frac{1}{2\epsilon} \left[\left(\frac{1}{2} + \epsilon \right)^n - \left(\frac{1}{2} - \epsilon \right)^n \right] \quad (14)$$

with the determinant $||\hat{\rho}_A(t)|| = \frac{1}{4} - \epsilon^2$. Note that we have written ρ_A^n in terms of ρ_A and the unity matrix. We could have done it via Cayley-Hamilton theorem[2], which states that any (square) matrix obeys its characteristic equation, i.e. a 2×2 matrix may be written as we did, as the square of the matrix (and any other power of it) may be related to the matrix and the unity matrix.

Then relation (6) may be proved by induction: for $n = 1$, we have

$$\hat{\rho}_B^2 = Tr_A\{\hat{\rho}(t)\hat{\rho}_A(t)\}. \quad (15)$$

We now assume it to be correct for $n = k + 1$

$$\hat{\rho}_B^{k+1} = Tr_A\{\hat{\rho}(t)\hat{\rho}_A^k(t)\} \quad (16)$$

and prove (6) for $n = k + 1$. By using (13) we can write

$$\hat{\rho}_B^{k+1} = \hat{\rho}_B^2 G(k) - \hat{\rho}_B ||\hat{\rho}_A(t)|| G(k-1) \quad (17)$$

Note that any power of $\hat{\rho}_B$ may be written in terms of $\hat{\rho}_B$ and $\hat{\rho}_B^2$ ($k \geq 2$). Multiplying the above equation by $\hat{\rho}_B$ we obtain

$$\hat{\rho}_B^{k+2} = \hat{\rho}_B^3 G(k) - \hat{\rho}_B^2 ||\hat{\rho}_A(t)|| G(k-1) \quad (18)$$

We obtain $\hat{\rho}_B^3$ from (6) as

$$\hat{\rho}_B^3 = Tr_A\{\hat{\rho}(t)\hat{\rho}_A^2(t)\} = Tr_A\{\hat{\rho}(t)[\hat{\rho}_A(t) + (\epsilon^2 - \frac{1}{4})\mathbf{1}]\} \quad (19)$$

where for the second equality we have used the Cayley-Hamilton theorem for the atomic density matrix: $\hat{\rho}_A^2(t) = \hat{\rho}_A(t) - ||\hat{\rho}_A(t)||\mathbf{1}$. Inserting (19) in (18) and after some algebra we find

$$\hat{\rho}_B^{k+2} = \hat{\rho}_B^2 G(k+1) - \hat{\rho}_B ||\hat{\rho}_A(t)|| G(k) \quad (20)$$

or

$$\hat{\rho}_B^{k+2} = Tr_A\{\hat{\rho}(t)\hat{\rho}_A^{k+1}(t)\} \quad (21)$$

that ends the prove of relation (6) by induction.

III. ATOMIC ENTROPY OPERATOR

With the tools we have developed up to here we can study the two-level atom-field interaction[3] or equivalently the ion-laser interaction [4, 5] and construct atom and field entropy operators. In the off-resonant atom-field interaction, i.e. the dispersive interaction, the (unnormalized) states $|c\rangle$ and $|s\rangle$ read[6]

$$|c\rangle = \frac{1}{\sqrt{2}}|\beta e^{-i\chi t}\rangle, \quad |s\rangle = \frac{1}{\sqrt{2}}|\beta e^{i\chi t}\rangle \quad (22)$$

where $|\beta\rangle = e^{-|\beta|^2/2} \sum_{k=0}^{\infty} \frac{\beta^k}{\sqrt{k!}}|k\rangle$ is the initial coherent state for the field, χ is the interaction constant. The atom was considered initially in a superposition of excited and ground $|\psi_A\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$.

We write the entropy operator as

$$\hat{S}_A = \ln \hat{\rho}_A^{-1} = \ln(1 - \hat{\rho}_A) - \ln \|\hat{\rho}_A\|, \quad (23)$$

such that the expectation value of \hat{S}_A is the entropy. In the above expression we have used that $\hat{\rho}_A^{-1}(t) = \hat{\xi}_A / \|\hat{\rho}_A(t)\|$ with the purity operator $\hat{\xi}_A = \mathbf{1} - \hat{\rho}_A(t)$. We can develop $\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ and use (13) to find

$$\hat{S}_A = F_1 \hat{\rho}_A + F_2 \mathbf{1} \quad (24)$$

with

$$\begin{aligned} F_1 &= \frac{1}{2\epsilon} \ln \left(\frac{1 - 2\epsilon}{1 + 2\epsilon} \right) \\ F_2 &= -\frac{1}{2} \left[\ln \|\hat{\rho}_A(t)\| + \frac{1}{2\epsilon} \ln \left(\frac{1 - 2\epsilon}{1 + 2\epsilon} \right) \right] \end{aligned} \quad (25)$$

Note that \hat{S}_A is linear in the atomic density matrix as expected from Cayley-Hamilton's theorem (for 2×2 matrices). From (24) we can calculate the atomic (field) entropy and the atomic (field) entropy fluctuations

$$\langle \hat{S}_A \rangle = F_2 + F_1(1 - 2\|\hat{\rho}_A(t)\|), \quad (26)$$

and

$$\langle \Delta \hat{S}_A \rangle = \sqrt{\langle \hat{S}_A^2 \rangle - \langle \hat{S}_A \rangle^2} = \ln \left(\frac{1 - 2\epsilon}{1 + 2\epsilon} \right) \|\hat{\rho}_A(t)\|^{1/2}. \quad (27)$$

IV. FIELD ENTROPY OPERATOR

We use the expression for $\hat{\rho}_B^n$ in terms of $\hat{\rho}_A$ to write the field entropy operator in terms of the atomic density operator

$$\hat{S}_B = Tr_A\{\hat{\rho}(t)\hat{S}_A(t)\hat{\rho}_A^{-1}(t)\} \quad (28)$$

Note that we can write the atomic entropy operator in terms of the operator used to define concurrence[7] because $\hat{\rho}_A^{-1}(t) = \frac{1}{\|\hat{\rho}_A(t)\|}\hat{\sigma}_y\hat{\rho}_A^*(t)\hat{\sigma}_y = \frac{\hat{\rho}_A(t)}{\|\hat{\rho}_A(t)\|}$ i.e.

$$\hat{S}_B = -\frac{1}{\|\hat{\rho}_A(t)\|}Tr_A\{\hat{\rho}(t)\hat{S}_A\hat{\rho}_A(t)\} \quad (29)$$

By inserting (24) into (28) we obtain

$$\hat{S}_B = Tr_A\{\hat{\rho}(t)(F_1 + F_2\hat{\rho}_A^{-1}(t))\} \quad (30)$$

using the expression of the inverse of the atomic density operator in terms of the purity operator, the entropy may be written as

$$\hat{S}_B = Tr_A\{\hat{\rho}(t)(F_1 + \frac{F_2}{\|\hat{\rho}_A(t)\|}[\mathbf{1} - \hat{\rho}_A(t)])\} \quad (31)$$

In terms of the field density matrix the field entropy operator is

$$\hat{S}_B = \left(F_1 + \frac{F_2}{\|\hat{\rho}_A(t)\|}\right)\hat{\rho}_B(t) - \frac{F_2}{\|\hat{\rho}_A(t)\|}\hat{\rho}_B^2(t) \quad (32)$$

from (5) we can write $\hat{\rho}_B^2(t)$ as

$$\hat{\rho}_B^2 = |c\rangle\langle c|\langle c|c\rangle + |s\rangle\langle s|\langle s|s\rangle + |c\rangle\langle s|\langle c|s\rangle + |s\rangle\langle c|\langle s|c\rangle. \quad (33)$$

From (22) we have that $\langle c|c\rangle = \langle s|s\rangle = 1/2$ and that $\langle c|s\rangle = \langle s|c\rangle^* = \exp(-|\beta|^2[1 - e^{2i\chi t}])/2$.

The entropy operator for the field is written as

$$\hat{S}_B = \left(F_1 + \frac{F_2}{2\|\hat{\rho}_A(t)\|}\right)(|c\rangle\langle c| + |s\rangle\langle s|) - \frac{F_2}{\|\hat{\rho}_A(t)\|}(\langle s|c\rangle|s\rangle\langle c| + \langle c|s\rangle|c\rangle\langle s|) \quad (34)$$

We have chosen the dispersive interaction so that all the terms forming the field density matrix and its square are coherent states, such that we can calculate the Wigner function associated to (32) in an easy way. We do it by means of the formula[8]

$$W_S(\alpha) = \sum_{n=0}^{\infty} (-1)^n \langle \alpha, n | \hat{S}_B | \alpha, n \rangle \quad (35)$$

where $|\alpha, n\rangle$ are the so-called displaced number states [9]. The explicit expression for the above equation is

$$\begin{aligned}
W_S(\alpha) = & e^{-2\beta^2 - 2|\alpha|^2 + 4\beta\alpha_x \cos \chi t} \left(F_1 + \frac{F_2}{2||\hat{\rho}_A(t)||} \right) \cosh(4\beta\alpha_y \sin \chi t) \\
& - e^{-2\beta^2 - 2|\alpha|^2 + 4\beta\alpha_x \cos \chi t} \frac{F_2}{2||\hat{\rho}_A(t)||} \cos(2\beta[\beta \sin 2\chi t - 2\alpha_x \sin \chi t]).
\end{aligned} \tag{36}$$

V. CONCLUSIONS

We have shown that the entropy operator for the subsystems may be obtained and in the case one of those subsystems is the quantized field. The connection with quasiprobability distribution functions [10, 11], in particular the association to the Wigner function for the entropy has been obtained.

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